The Seventh Annual CSU Freshman–Sophomore Mathematics Competition

Wednesday, April 17, 2013
6:00–7:45 PM in RT 1516

(1) Peter writes a random two-digit number. What are the chances that the sum of its digits equals 13?

(2) Does there exist a continuous function which takes *every* real value exactly three times? (If yes, give a sketch of the graph and explain why the conditions are satisfied. If no, explain why.)

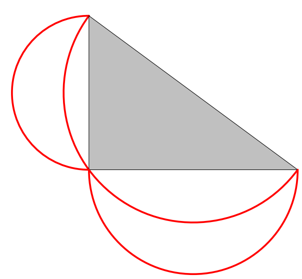
(3) There are 100 lamps on the dashboard and 100 corresponding switches. On the first day Ben switched all the lamps on. On the next day he switched off every second lamp (so now half of the lamps are on). On the third day he changed the position of the switch for every third lamp (e.g. the lamp number 3 is now off, but the lamp number 6 is back on). On the fourth day he changed the position of the switch for every fourth lamp, and so on. Determine which lamps will be on after the 100th day. Provide explanation.

(4) Evaluate the following integral:

$$\int_0^{\pi/2} \left(\sin^2(\sin x) + \cos^2(\cos x)\right) dx.$$

(Hint: $\cos x = \sin(\pi/2 - x)$.)

(5) The area of the right triangle below equals 5 square inches. Find the total area of the "moons" bounded by the red semicircles.



(6) In a Long Division class Peter was given a polynomial $p(x)$ and was asked to divide it by $x - 1$ with remainder. Peter got remainder 2013. Then he was asked to divide $p(x)$ by $x - 2013$ with remainder. This time Peter got remainder 1. What remainder would Peter get if he is asked to divide $p(x)$ by $(x-1)(x-2013)$? (Remark: Peter never makes mistakes in long division!)

(7) Ben has six metal rods of different length which he can use to make a tetrahedron. Can Ben use the same rods to make two triangles instead? Explain your answer.

(8) Peter and Ben play a game on an $m \times n$ grid of squares. Peter has a red crayon and Ben has a blue crayon. They take turns picking a square (of any size they want) along the grid lines and coloring it in their color. Whoever colors the last square wins. If Peter goes first, does he have a winning strategy? Answer this question for a (a) 3×5 grid, (b) 13×100 grid, (c) 2×5 grid. Provide explanation.