Elliptic Spaces with maximal toral rank

(Work in progress w/ Mark Walker.)

"Hilali conj."

REFERENCES

1. Manuel Amann, *A note on the Hilali conjecture*, Forum Math. **29** (2017), no. 2, 251–257. MR 3619111
2. Stephen Halperin, *Finiteness in the minimal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173–199. MR 461508
3. ———, *Rational homotopy and torus actions*, Aspects of topology, London Math. Soc. Lecture Note Ser., vol. 93, Cambridge Univ. Press, Cambridge, 1985, pp. 293–306. MR 787835
4. Mark E. Walker, *Total Betti numbers of modules of finite projective dimension*, Ann. of Math. (2) **186** (2017), no. 2, 641–646. MR 3702675
5. ———, *Total betti numbers of modules of finite projective dimension*, arXiv:1702.02560v2 [math.AC], 2017.

for "elliptic spaces"

Toral rank conj

algebraic results

(4. is a v. condensed version of 5.)

Walker:

Related to the "B-E-H" conjecture (see 4.)

Theorem 1. *Let R be a commutative Noetherian ring such that* Spec(R) *is connected and let M be a non-zero, finitely generated R-module of finite projective dimension. Assume either*

(1) *R is locally a complete intersection*[1] *and M is 2-torsion free, or*
(2) *R has characteristic p for an odd prime p.*

Then for any finite projective resolution

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M we have

$$\sum_i \operatorname{rank}_R(P_i) \geq 2^c,$$

where $c = \operatorname{height}_R(\operatorname{ann}_R(M))$, the height of the annihilator ideal of M.

THEOREM 2. *Assume $(R, \mathfrak{m}, k)$ is a local (Noetherian, commutative) ring of Krull dimension d and that M is a nonzero R-module of finite length and finite projective dimension. If either*

(1) *R is the quotient of a regular local ring by a regular sequence of elements and 2 is invertible in R, or*
(2) *R contains $\mathbb{Z}/p$ as a subring for an odd prime p,*

then $\sum_i \beta_i(M) \geq 2^d$.

Moreover, if the assumptions in (1) hold and $\sum_i \beta_i(M) = 2^d$, then M is isomorphic to the quotient of R by a regular sequence of d elements.

From pre-print:

involved in proving Th. 1 + Th. 2 above.

With suitable modifications, the proof of Theorem 6 establishes the following analogue for dg-modules over graded rings:

Theorem 7. *Let R be the (cohomologically) graded ring $k[t_1, \ldots, t_d]$ where k is a field with $\mathrm{char}(k) \neq 2$ and $t_1, \ldots, t_d$ are variables of strictly positive, even degree. If F is a semi-free dg-R-module having non-zero homology of finite dimension over k, then*

$$\sum_i \dim_k H_i(F \otimes_R R/(t_1, \ldots, t_d)) \geq 2^d \cdot \frac{|\sum_i (-1)^i \dim_k(H_i(F))|}{\sum_i \dim_k(H_i(F))}.$$

Then, to topology (Total rank conj. vs. Toral rank conj.)

As an application of Theorem 7, we address the rational cohomology of spaces admitting (almost) free torus actions. Let T be a torus of dimension d,

$$T = \overbrace{S^1 \times \cdots \times S^1}^{d},$$

regarded as a topological group. One says that T acts *(almost) freely* on a space X if there is a continuous action of T on X such that the stabilizer of each point of X is a finite subgroup of T. The *Toral Rank Conjecture* of Halperin [28] predicts that if T acts almost freely on a simply connected, compact CW complex X, then

$$\underbrace{\sum_j \dim_{\mathbb{Q}} H^j(X, \mathbb{Q})}_{\text{"total rank } h(X)\text{"}} \geq 2^d.$$

(free if trivial.)

note:
$\dim(H^*(T;\mathbb{Q})) = 2^d$

an exterior algebra on d generators of degree 1.

more from preprint.

This ratio is ≤ 1 and is often zero.

The following consequence of Theorem 7 represents partial progress towards a proof of the Toral Rank Conjecture.

Corollary 8. *Suppose a d-dimensional torus T acts (almost) freely on a simply connected, compact CW complex X and let X/T denote the quotient space. Then*

$$\sum_i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \geq 2^d \cdot \frac{|\sum_i (-1)^i \dim_{\mathbb{Q}} H^i(X/T, \mathbb{Q})|}{\sum_i \dim_{\mathbb{Q}} H^i(X/T, \mathbb{Q})}.$$

$$\frac{|\chi(X/T)|}{h(X/T)}$$

In particular, if the non-zero rational cohomology of X/T is concentrated in even degrees, then the Toral Rank Conjecture holds for X.

As an example, suppose X is a simply connected, rationally elliptic space; the latter condition means that $\sum_q \dim_{\mathbb{Q}} \pi_q(X)_{\mathbb{Q}} < \infty$, where $\pi_q(X)_{\mathbb{Q}}$ denotes the q^{th} rational homotopy group of X. Suppose also that a d-dimensional torus T acts almost freely on X and assume that

$$\chi_\pi(X) := \sum_i (-1)^i \dim_Q \pi_i(X)_{\mathbb{Q}} = -d.$$

(This is the largest possible value: in general, given such a torus action on such a space X, one has $\chi_\pi(X) \leq -d$; see [24, 7.13].) It follows [24, 2.75] that the rational cohomology of X/T is concentrated in even degrees and thus, by Corollary 8, the Toral Rank Conjecture holds in this situation.

for elliptic X, toral rank is no larger than $-\chi_\pi(X)$, (see below).

Direct Results (not following from algebraic results.)

Proposition 1.1. *Let X be an elliptic space with homotopy Euler characteristic $\chi_\pi(X) = -n$. If the toral rank of X is n, then the minimal model $(\wedge V, d)$ of X is two-stage (pure) and satisfies $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subseteq \wedge(V^{\text{even}})$.*

as in Mark's ex.: toral rank is maximal.

i.e., an elliptic space with maximal toral rank is of a very specialized form.

With this observation made, it is possible to obtain the corollary above directly (and in most cases to improve the lower bound on the cohomology by a factor of 2 at the same time).

Proposition 1.2. *Let X be an elliptic space with minimal model $(\wedge V, d)$ a two-stage (pure) model that satisfies $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subseteq \wedge(V^{\text{even}})$. Suppose we have $\dim(V^{\text{even}}) = k$ and $\dim(V^{\text{odd}}) = k+n$, for some $n \geq 0$, so that $\chi_\pi(X) = -n$. If $k = 0$ (so that X has only odd-degree non-zero rational homotopy), then we have $\dim(H^*(X;\mathbb{Q})) = 2^n$. Otherwise, i.e., if $k \geq 1$, we have $\dim(H^*(X;\mathbb{Q})) \geq 2^{n+1}$.* $= 2 \cdot 2^n$

Elliptic: minimal model $\mathcal{M}_X = \Lambda V, d$

with V f.d. and $H(\Lambda V, d)$ f.d.

Then $\mathcal{M}_X = \Lambda(\underbrace{u_1, \dots, u_k}_{k \text{ even degree}}, \underbrace{v_1, \dots, v_{n+k}}_{n+k \text{ odd degree}}), d$

Fact: elliptic space entails no more even gens. than odd gens.

Then $\chi_\pi = -n = -$("excess odd degree gens. over even deg. gens.")

And toral rank is at most n. $(= -\chi_\pi)$

So toral rank conj. predicts $h(X) \geq 2^n$ under hypotheses of Prop. 1.2

- Proof is by a straightforward S.S. argument.

- Also: Hilali conjecture: Elliptic space: $h(X) \geq$ total dim. of $\pi_*(X) \otimes \mathbb{Q}$. $=$ # of gens. in $\mathcal{M}_X$.

Proposition 1.3. *Let X be an elliptic space with minimal model $(\wedge V, d)$ a two-stage (pure) model that satisfies $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subseteq \wedge(V^{\text{even}})$. Suppose we have $\dim(V^{\text{even}}) = k$ and $\dim(V^{\text{odd}}) = k+n$, for some $n \geq 0$, so that $\chi_\pi(X) = -n$. If $k = 0$ (so that X has only odd-degree non-zero rational homotopy), then we have $\dim(H^*(X;\mathbb{Q})) = 2^n$. Otherwise, i.e., if $k \geq 1$, we have $\dim(H^*(X;\mathbb{Q})) \geq 2(k+n) = \dim(\pi_*(X) \otimes \mathbb{Q}) + n$.*

$\underbrace{\dim(\pi_*(X)\otimes\mathbb{Q})}_{(k+k+n)} + n$ = # of generators $+ n$

(Proof of this uses a straightforward "Gysin sequence" argument.)

$\underline{\text{Exs.}}$ $\mathcal{M}_X = \Lambda(\underbrace{\overset{2}{u_1}, \overset{2}{u_2}}_{d=0}, \overset{3}{v_1}, \overset{3}{v_2}, \overset{3}{v_3})$.

$v_1 \to u_1^2$, $v_2 \to u_1u_2$, $v_3 \to u_2^2$

total rank = 1

# of gens = 5.

$\chi_\pi = -1$

(in Props. 1.2, 1.3, $k=2$, $n=1$)

$H^*(X;\mathbb{Q}) =$

7	
6	
5	$\langle v_1u_2 - v_2u_1\rangle$, $\langle v_2u_2 - v_3u_1\rangle$
4	
3	
2	u_1 u_2
1	
0	1

$h(X) = 6$ TRC $\geq 2^1$ Hilali ≥ 5

(predictions from conjectures)

$\underline{\text{Ex.}}$ $\mathcal{M}_X = \Lambda(v_1, \ldots, v_n)$, $d=0$ with $|v_i|$ odd.

(in Props. 1.2, 1.3, $k=0$, $n=n$)

Then $h(X) = 2^n$. TRC: $\geq 2^n$ (sharp)

Hilali: $\geq n$ (low ball)

$\underline{\text{Ex.}}$ $\mathcal{M}_X = \Lambda(u_1, \ldots, u_k, v_1, \ldots, v_k)$, d

$|u_i|$ even

$|v_i| = 2|u_i| - 1$ and $dv_i = u_i^2$ (X is a product of even-dimensional spheres)

(in Props. 1.2, 1.3, $k=k$, $n=0$).

$h(X) = 2^k$. TRC: $2^0 = 1$ (low ball!)

Hilali: $2k$. (low ball)