DIGITAL TOPOLOGY: SPECULATIONS TOWARDS APPLICATIONS

GREGORY LUPTON, JOHN OPREA, AND NICHOLAS A. SCOVILLE

The body of work so far:

REFERENCES

[1] G. Lupton, J. Oprea, and N. Scoville, *A fundamental group for digital images*, J. Appl. Comput. Topol. **5** (2021), no. 2, 249–311. MR 4259435

[2] ———, *Homotopy theory in digital topology*, Discrete Comput. Geom. **67** (2022), no. 1, 112–165. MR 4356246

[3] ———, *Subdivision of maps in digital topology*, Discrete Comput. Geom. **67** (2022), no. 3, 698–742. MR 4393077

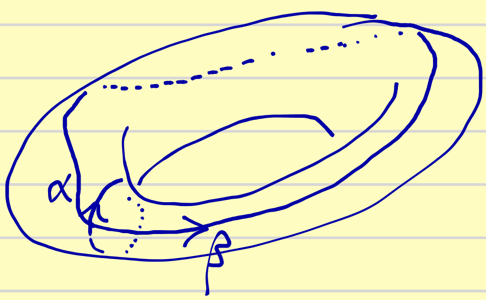
[4] G. Lupton and N. Scoville. *Digital fundamental groups and edge groups of clique complexes.* MS of 22 pp.

JApCT?

Digital Fundamental Group

In ordinary topology $\qquad X$

Consider loops in a topological space X



$$I \underset{\beta}{\overset{\alpha}{\rightrightarrows}}$$

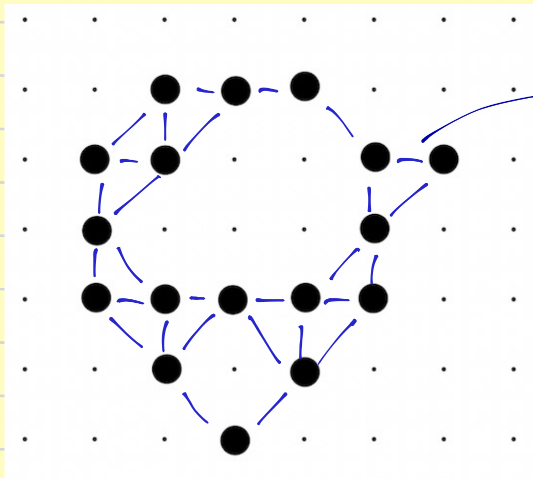
Fundamental group $\pi_1(X) = \{\alpha : I \to X\} / \sim$

Motto: $\pi_1(X)$ detects "1-dimensial holes" in X

In digital topology

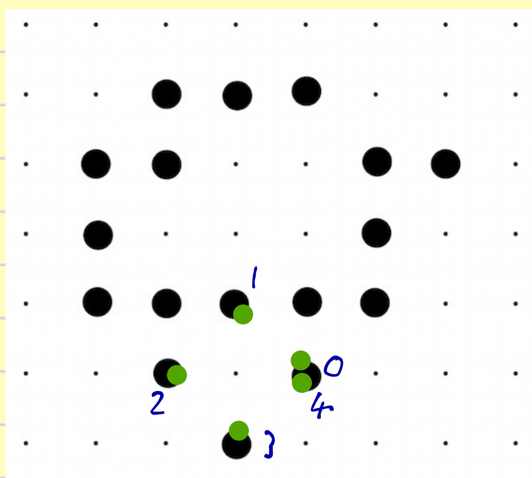
- use loops in a digital image.

digital image X (graph).



"adjacencies"

loop



X

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \xrightarrow{\alpha}$$

$I_4 \subseteq \mathbb{Z}$

all lengths N

$$\pi_1(X) = \{ \alpha : I_N \longrightarrow X \} / \sim$$

"digital fundamental group"

Equivalence relation allows for equivalence of loops of different length.

Many calculations flow from:

S-VK

a result of [4]

Theorem (Digital Seifert-van Kampen). *Let U and V be connected digital images in some $\mathbb{Z}^n$ with connected intersection $U \cap V$. Choose $x_0 \in U \cap V$ for the basepoint of $U \cap V$, U, V, and $U \cup V$. If U and V have disconnected complements, then*

$$\begin{tikzcd} \pi_1(U \cap V; x_0) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi_1(U; x_0) \arrow[d, "\psi_1"] \\ \pi_1(V; x_0) \arrow[r, "\psi_2"'] & \pi_1(U \cup V; x_0) \end{tikzcd}$$

$X = U \cup X$

$\pi_1(X)$

is a pushout diagram of groups and homomorphisms, with i_1, i_2, ψ_1 and ψ_2 the homomorphisms of fundamental groups induced by the inclusions $U \cap V \to U$, $U \cap V \to V$, $U \to U \cup V$ and $V \to U \cup V$ respectively.

"vee"

e.g. $X = U \vee V$ $\qquad \pi_1(X) = \pi_1(U) * \pi_1(V)$

"one-point union" $\qquad$ free product gps.

$U \cap V = \{*\}$ $\qquad \pi_1(U \cap V) = \{e\}$.

also from [4]

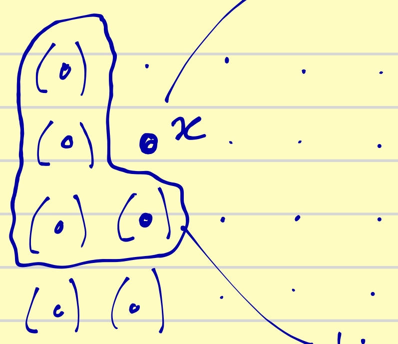
Theorem. *Let $X \in \mathbb{Z}^2$ be a connected 2D digital image. Then $\pi_1(X; x_0)$ is a free group.*

Idea of proof. - induct on # pts. of X

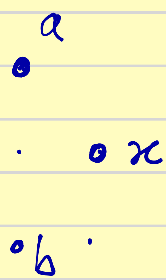
- use S-VK.

Say $x \in X$ is "maximal" in X (lex ordering)

(•) = a point that may be in X



One case:



$\text{link}(x) = \{a, b\}$ as shown.

Assume
$P(a,b)$ is a path in $X - \{x\}$ from $a \leftrightarrow b$.

Set $U = P(a,b) \cup \{x\}$ (WLOG a cycle graph / circle).
so $\pi_1(U) \cong \mathbb{Z}$. (a result of [1] and/or [4])

Set $V = X - \{x\}$.

$U \cap V = P(a,b)$ – "contractible" with $\pi_1(U \cap V) = \{e\}$

So S-VK:

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V)$$

$$\cong \mathbb{Z} * \text{free} = \text{free}.$$

by ind. hyp.

– other cases complete the induction.

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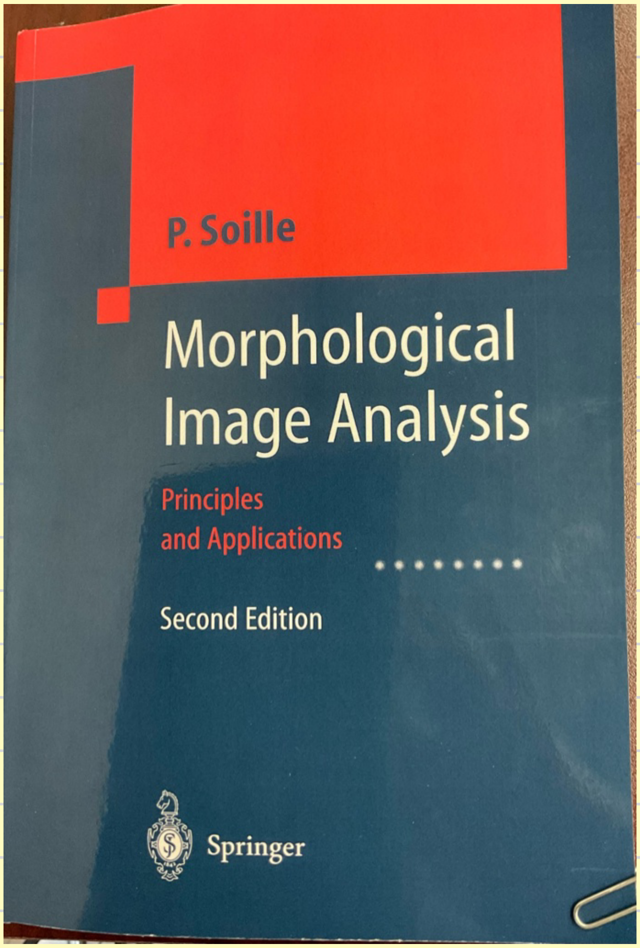
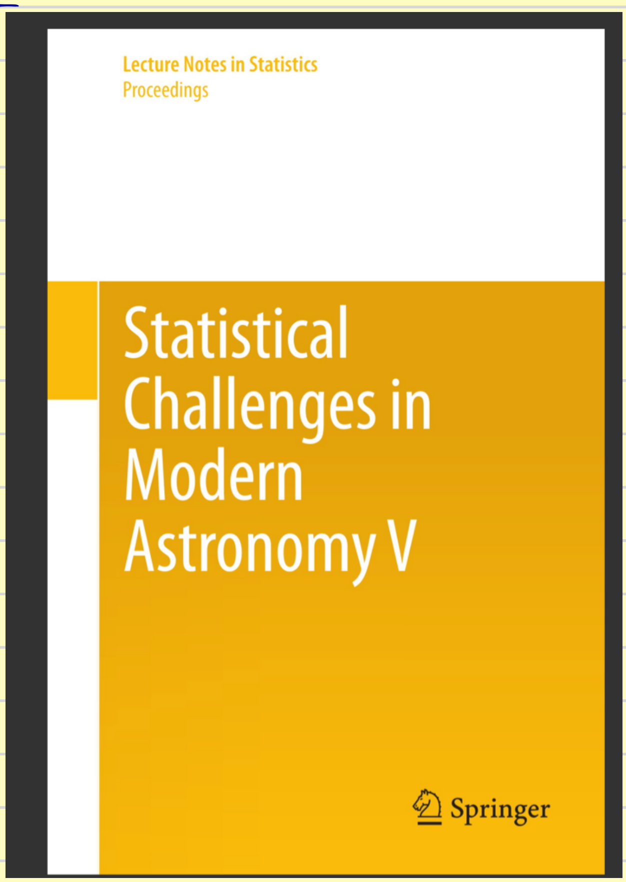
Towards applications (Directions for further work).



- an area of image analysis that gives many methods of geometric/topological feature extraction.

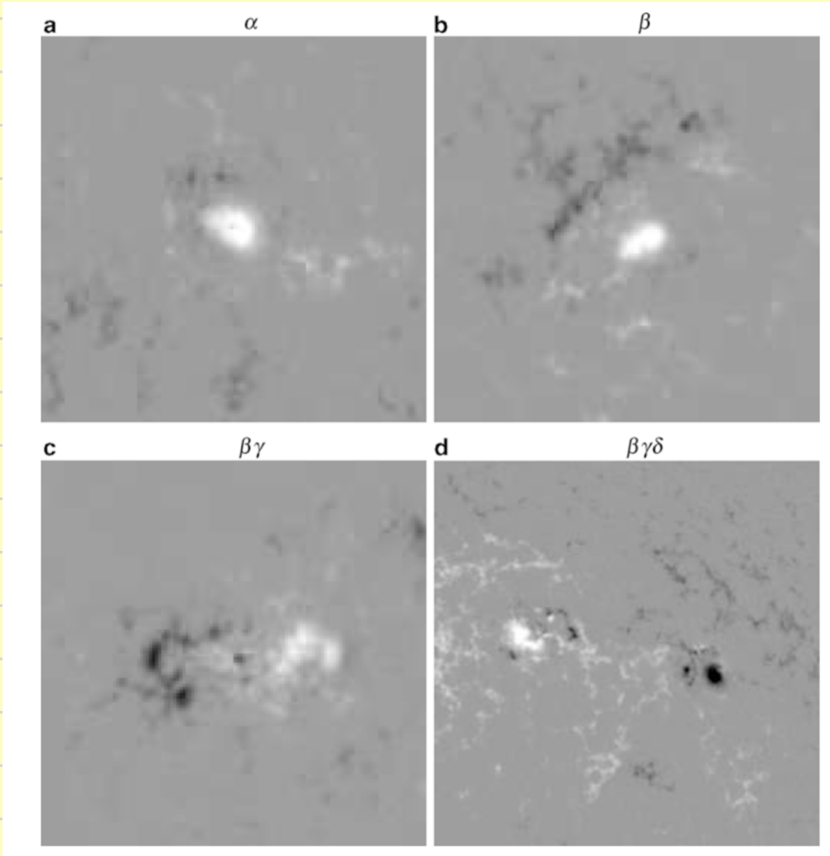
Ex.



Chapter 31
Morphological Image Analysis and Sunspot Classification

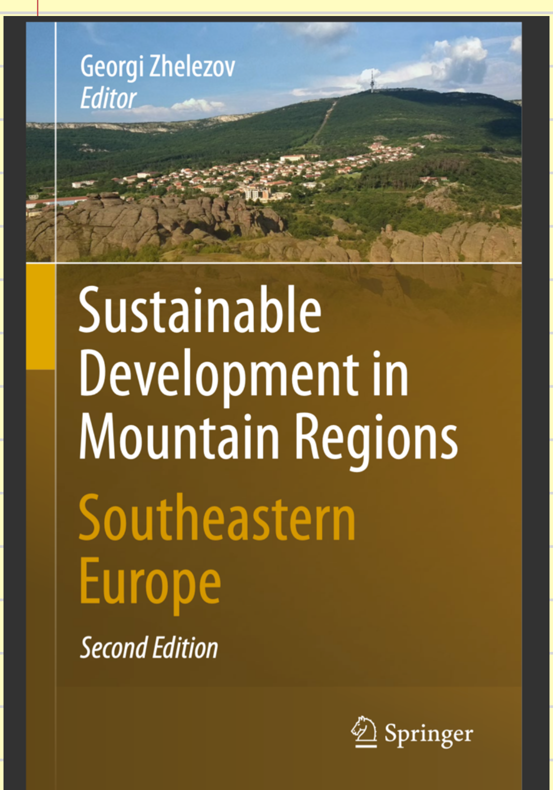
David Stenning, Vinay Kashyap, Thomas C.M. Lee, David A. van Dyk, and C. Alex Young

Abstract The morphology of sunspot groups is predictive both of their future evolution and of explosive associated events higher in the solar atmosphere, such as solar flares and coronal mass ejections. To aid in this prediction, sunspot groups are manually classified according to one of a number of schemes. This process is both laborious and prone to inconsistencies stemming from the subjective nature of the classification. In this paper we describe how mathematical morphology can be used to extract numerical summaries of sunspot images that are relevant to their classification and can be used as features in an automated classification scheme. We include a general overview of basic morphological operations and describe our ongoing work on detecting and classifying sunspot groups using these techniques.



digital images of sunspot patterns.

Ex.



Chapter 11
Mapping Forest Fragmentation Based on Morphological Image Analysis of Mountain Regions in Bulgaria and Slovakia

Rumiana Vatseva, Monika Kopecka, and Jozef Novacek

Abstract Forest landscapes are at high risk of fragmentation as a result of changes in land cover and land use, which affect habitat loss and degradation. Therefore, it is essential in the forest management and biodiversity policy context to monitor and assess forest fragmentation using reliable data from remote sensing and GIS. This chapter focuses on the assessment and mapping forest fragmentation in two mountain regions in Bulgaria (part of the Eastern Rhodopes Mountain) and in Slovakia (the Tatra Mountains). The aim is to point out the correlation between the observed land cover changes during the 22-year period from 1990 to 2012 and forest fragmentation, which affects loss of biodiversity. The landscape fragmentation tool (LFT v2.0) was used to map the forest fragmentation and to analyze the forest pattern. The results indicate more significant forest fragmentation in the Tatra Mountains and decrease of the compact forest areas (i.e., core forest) in both mountain regions. The main causes for forest fragmentation were natural disasters and human activities. Generated maps identify areas in which to focus management efforts aimed at minimizing forest fragmentation.

Keywords Forest fragmentation • Morphological image analysis • Land cover • Land use • Eastern Rhodopes Mountain in Bulgaria • Tatra Mountains in Slovakia

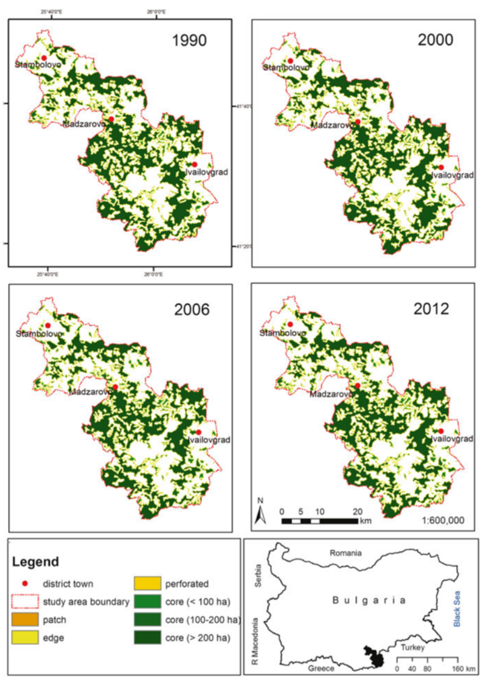


Fig. 11.3 Maps of forest fragmentation in the Eastern Rhodopes Mountain in Bulgaria for the period 1990–2012

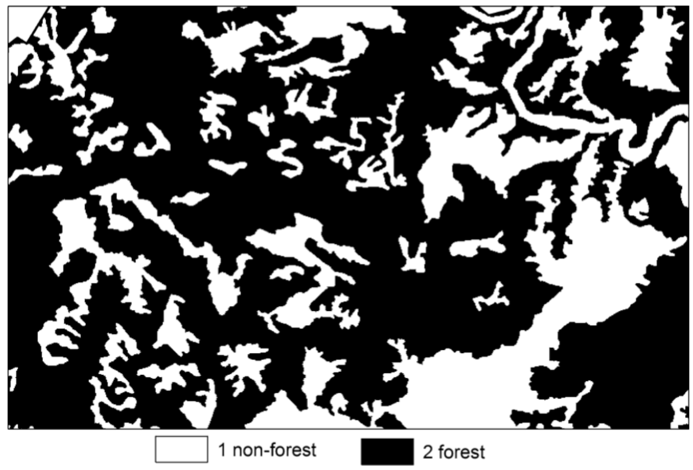


Fig. 11.2 Binary forest map derived from the CORINE land cover data

We use $\pi_1(X)$ but homology groups like $H_1(X)$ have great potential, too.

Further references

[5] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational homology*, vol. 157 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

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[9] J. Peters and P. Wasilewski. Tolerance spaces: origins, theoretical aspects and applications. *Information Sciences*, 195:211–225, 2012.

very close to digital images.

Potential collaborator (Computer/Electrical Engineer @ U. of Manitoba).

Gloss on the further references:

[7]—MIA is used to scientifically classify or aggregate visually different images of sunspot patterns. The resulting classification is then used to study and predict solar phenomena such as solar flares.

[8]—MIA is used on images of forested regions to develop an index of "fragmentation" that is suitable for methodical (scientific) comparison of whether forests are more or less "fragmented." Then this index of fragmentation is used in a study of ecological effects of fragmentation.

In both [7] and [8], the first step is to develop a classification of images (of sunspots, of forested regions) based on certain geometric or topological features.

Finally, [9] contains an application in which some ideas from tolerance space theory are used to classify images of hands—frames from video of hand motions—with a view towards determining arthritis in the subject. The classification of images here is based on notions of distance between two images.